A RANDOM WALK AT INFINITY

A Layperson's Exposition of
Kumar Eswaran's Proposed Proofs of the
Riemann Hypothesis and the
Generalised Riemann Hypothesis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\zeta(s) = 0$

$\mathcal{R}e(s) = \frac{1}{2}$

VINAYAK ESWARAN

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by

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This book presents the problem of the Riemann Hypothesis (and later the Generalised Riemann Hypothesis). It details the relevant developments from the work of Riemann (1859) to Littlewood (1912) and gives the comprehensive proofs proposed by Kumar Eswaran (2016-18) that use the probabilistic *Law of the Iterated Logarithm* of A.Y. Khinchin and A.N. Kolmogorov from the 1920's. No mathematics discovered after 1930 is used in the proofs.

The implications of the simultaneous proof of these two problems, considered to be the greatest in Pure Mathematics, are staggering in ways totally unanticipated. Foremost among these is that, contrary to universal expectation, the proofs say nothing at all new about prime numbers. Further, while they speak for the logical consistency of Analysis beyond any result hitherto known, they also question the foundational promise and future prospects of Analytic Number theory.

The book assumes no prior knowledge of the Riemann Hypothesis, or even advanced complex analysis. Starting from first principles, and free of jargon and abstract notation, **the book can be read by undergraduate students** of any mathematical discipline of science and engineering.

While its tone and language are directed at STEM educated laypersons, the book **contains the detailed complete proofs**. So mathematicians, scientists and graduate students are invited to read the book and critique the proofs (after reading, at least, Chapters 5 to 12 in their entirety).

From the book jacket

The **Riemann Hypothesis**, conjectured by Bernard Riemann in 1859, is considered the greatest unsolved problem in pure mathematics. In 2016, Kumar Eswaran, a theoretical physicist and professor of computer science, uploaded a proposed proof on his website. Despite thousands of downloads, prestigious mathematics journals refused even to formally review the proof, let alone publish it.

In this book, Vinayak Eswaran traces the problem from its Riemannian origin to its final proof, using mathematics an undergraduate in science or engineering can follow. As it contains the full proof with all details, mathematicians and scientists can critique the proof if they choose.

The proof is remarkable because it uses Probability Theory, very far from the Complex Analysis roots of the original problem. Even more remarkably, it uses no theorem after 1930. Crucial to the proof is the **Liouville integer function $\boldsymbol{\lambda}(n)$** which has values of $+1$ or -1, depending on whether the natural number n has an even or odd number of prime factors. The proof hinges on showing $\boldsymbol{\lambda}(n)$ behaves increasingly like a perfect coin-toss (with equal ± 1 probability & sequentially independent values) as n increases, $n{=}1, 2, \ldots \infty$, so that the sequence becomes a random walk "at infinity". However, the proof is not probabilistic but classical, with traditional mathematical certainty.

Most remarkably, the **Generalised Riemann Hypothesis**, previously presumed to be a far more difficult problem, is proved as a corollary in a double proof that will shake the foundations of analytic number theory.

Written without jargon in a plain but lucid style, this book brings out the beauty of the proofs and their profound and unexpected implications. It should be read by anyone interested in higher mathematics.

An extended extract including the **first seven chapters** is downloadable (free or nominally charged) from:
`https://leanpub.com/A_Random_Walk_at_Infinity_extract`
The extract comprises the Preface, Contents, etc., and the first 7 chapters of the print book culminating in the proof of the 'Eswaran' Conjecture. A special case of the 1965 Chowla Conjecture (an open problem in analytic number theory) is shown to follow as its corollary. The extract ends just a step away from the final proof of the Riemann Hypothesis by invoking the probabilistic *Law of the Iterated Logarithm*. The abstracts of the remaining technical chapters can be accessed from its Contents page.

Summary of the proofs

The proposed proof of the Riemann Hypothesis follows from the proof of an Equivalent Statement which would be true if the Hypothesis is true and false if the Hypothesis is false. This Equivalent statement is:

$$\boxed{\lim_{N\to\infty} \frac{L(N)}{N^{\frac{1}{2}+\varepsilon}} = 0, \text{ for } \varepsilon > 0} \qquad \text{(Eq\_RH)}$$

where $L(N) \equiv \sum_{n=1}^{N} \lambda(n)$ is the **summatory Liouville function** and $\lambda(n)$ is the **Liouville integer function** which has the value $+1$ if the natural number n has an even number of prime factors and -1 for an odd number of prime factors. The statement is well-known, analogous to one derived by J.E. Littewood in 1912 for the closely-related Möbius integer function.

The premise Eq_RH follows directly from the proof of the Conjecture:
The Liouville integer function $\lambda(n)$, sequentially evaluated, from $n = 1, 2, ...$, behaves like a perfect coin-toss in any finite neighbourhood of n, as $n \to \infty$, by having (a) Equal Probabilities of ± 1 values, and being (b) Independent of any finite number of preceding λ's.

The argument is **not** that $\lambda(n)$ is, or that it is analogous to, a random coin-toss, but rather that it can be **proved** to behave like a perfect coin-toss as $n \to \infty$ when considered *en masse* and *sequentially*. The deterministic nature of $\lambda(n)$ is used to establish that the conditions (a) and (b) in the conjecture above apply as $n \to \infty$. This makes the sum $L(N)$ essentially behave like a specific case of a Random Walk "at infinity".

This result, coupled with the fact that the limit in Eq_RH is also decided "at infinity", allows us to use the *Law of the Iterated Logarithm* (LIL) to prove the equivalent statement — which also proves the Riemann Hypothesis. The use of this famous theorem from Probability Theory is justified because the *deterministic* $\lambda(n)$ is shown to satisfy the two conditions for a *random* coin-toss under the stated conditions. However, for reasons that become clear in the book, the final proof is **not probabilistic**, but has canonical mathematical certainty.

An astonishing corollary of the proof of the Riemann Hypothesis is the proof of the **Generalised Riemann Hypothesis (GRH)**, which is first shown to have an analogous equivalent statement:

$$\boxed{\lim_{N\to\infty} \frac{S_L(N,\chi)}{N^{\frac{1}{2}+\varepsilon}} = 0, \quad \text{for } \varepsilon > 0} \qquad \text{(Eq\_GRH)}$$

with $S_L(N,\chi) \equiv \sum_{n=1}^{N} \chi(n)\,\lambda(n)$, where $\chi(n)$ is the relevant *Dirichlet character* for the particular L-function under consideration.

The proof of Eq_GRH follows from the already-proved coin-toss behaviour of the $\lambda(n)$'s "at infinity", coupled with the cyclical nature of the Dirichlet character $\chi(n)$, completing the proof of the GRH in its most general form.

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